OPTIMIZATION OF MARITIME TRANSPORT MANAGEMENT

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Abstract

In the present paper different methods and strategies for optimizing the maritime shipping management are revealed. The beginning of the article is dedicated to some opportunities for constructing the mathematical model of the problem. Then an approach to determining its solution is presented, observing all the positive points concerning its level of difficulty with different aspects of dynamic optimizing and duality included.

Maritime transport is an essential component of the global logistics system as its role is not only to effect the physical movement of goods but also to provide a far broader range of logistics services. It can be defined as a key logistics function related to the relocation of goods by certain means of transport, such as the various types of ships in the supply chain and in particular, the loading, shipping and unloading of the goods from and to the respective ports (Estache and Trujillo, 2009). The advantages of this mode of transport lie in the high carriage capacity and the virtually unlimited carrying capacity allowing large-scale intercontinental carriage of goods, and especially in the extremely low cost (Ducruet and Notteboom, 2011). From the above it becomes clear that this mode of transport connects quite remote destinations within the logistics system of the maritime transport and plays a key role in connecting many of the participants in the logistics processes (Bowersox, Closs and Cooper, 2002): manufacturers, suppliers, customers, consumers, warehouses, etc. If maritime transport is not included and above all integrated into the logistics processes, this will inevitably incur additional costs, unwanted delays, increased risk of accidents or other disturbances in the logistics processes (Caramia and Guerriero, 2009). In this context, maritime transport is to a great extent responsible for the carriage of goods in a way that is to the highest degree synchronized with the other components of the logistics process (Fransoo and Lee, 2011).

The enormous importance of maritime transport can be inferred from the ever increasing rate of world trade realized by seaborne shipments (Figure 1). The figure

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below shows clearly the essential role of maritime transport in the context of World Trade, GDP and Industrial Production Index.


Figure 1. Comparison between the rates of world trade, seaborne trade, GDP and IPI

The graphs in Figure 1 show that after the recovery of the global economy from the recession, seaborne shipments have grown steadily, especially over recent years. In this regard it should be noted that transport operators are increasingly faced with the problem of how to manage the flow of goods along the various routes for maximum benefit (Wilmsmeier and Hoffmann, 2008), i.e. to minimize the prime costs and expenses for freight transportation. In this connection, the present study aims to construct an idealized economic-mathematical model for optimization of the costs incurred by transport operators for managing the different route lines: a solution with both regional and global application. One of the important issues to be resolved by such an operator is to decide on how many and what ship types of the available fleet
to use, to operate different routes in the best possible way. This requires not only meeting the needs of the operator’s customers, but also minimizing the overall costs of freight transportation along the respective routes.

Therefore we think that the economic-mathematical model proposed in this study aims to help maritime transport operators solve this problem. From a practical point of view, this problem can be solved most easily by using online computational software (for example www.zweigmedia.com and www.wolframalpha.com), which nowadays is a convenient and highly recommended approach.

Along with the mathematical formalization of the problem, an attempt will be made to propose suitable approaches and corresponding methods to find the optimal plans for the relevant mathematical models. Here we can argue that a certain mathematical model may be perfect in terms of the formalization it achieves, but if there is no method to make it feasible, it will remain just a human creation with no practical value.

As is well known (Atanasov, 2015), the initial stage of modeling is formulation and analysis of the economic problem at hand. Let us assume that a maritime operator can operate $m$ number of routes, and let the given future period be sufficiently foreseeable so that the company has complete clarity regarding the quantity of loads to be transported. Principally, this assumption does not impair the generality of the problem discussed, given that present-day operators operate under clear contractual relationships with relatively long durations, and very rarely, only in certain exceptional circumstances, make emergency decisions (Fremont, 2009). Under these conditions, the operator must decide how many and what type of vessels (e.g. container ships) to allocate to each route, in order to simultaneously achieve customer satisfaction and minimize the cost of freight transportation.

Let us further assume that the container ships available to the operator can be subdivided into $n$ number of types. As already mentioned, in terms of maritime practice it is justified to group the vessels by tonnage and capacity (Slack, 2011). Furthermore, we should take into account the fact that not every type of container ships can be allocated to each of the operated routes, as some of the destination ports do not allow mooring such types of vessels (e.g. draft, room for maneuver, etc.).

For the purpose of constructing the mathematical model of the problem we will introduce the following symbols:

$$c_{ij}: \text{the expenses for a complete run of a container ship of type } j \text{ along route } i, \quad i = 1, m, \quad j = 1, n;$$
\[d_{ij} : \text{the number of days a container ship of type } j \text{ along route } i \text{ waits lying at anchor at a roadstead, } i = 1, m, \quad j = 1, n;\]
\[r_j : \text{the expenses for a one-day wait at roadstead of a container ship of type } j, \quad j = 1, n;\]
\[f_{ij} : \text{the number of container ships of type } j \text{ whose technical characteristics allow completion of route } i, \quad i = 1, m, \quad j = 1, n;\]
\[\lambda_{ij} : \text{the container carrying capacity of a ship of type } j \text{ on route } i, \quad i = 1, m, \quad j = 1, n;\]
\[a_i : \text{the total number of containers to be shipped along route } i, \quad i = 1, m;\]
\[b_j : \text{the number of container ships of type } j \text{ available to the operator, } j = 1, n.\]

It should be noted, that the above parameters are governed by the following mathematical correlation:

\[f_{ij} \leq \min \left\{ \left[ \frac{a_i}{\lambda_{ij}} \right], b_j \right\},\]

where \( \left[ \frac{a_i}{\lambda_{ij}} \right] \) constitutes the entire part of \( \frac{a_i}{\lambda_{ij}} \).

To complete the economic-mathematical model of the problem at hand we will further introduce the variables \( x_{ij} \), denoting the unknown number of container ships of type \( j \) to be allocated on route \( i \), \( i = 1, m, \quad j = 1, n \).

The target function, whose minimum value is the object to be found, includes the total cost of container transportation to the end users. These costs are represented by two items: the costs incurred in the operation of the container ships during transport, and the costs of lying at roadstead. According to the symbols adopted, the first item is represented as follows:

\[\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij},\]

while the second item, i.e. the costs for lying at roadstead take the following mathematical expression:
When the above two groups of costs are added together, the optimality criterion, whose minimum value will be sought, will have the following expression:

\[ Z(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} r_{j} d_{ij} x_{ij}. \]

What could be the possible restrictions that would give rise to certain constraints? First, the containers scheduled for shipping must be transported, i.e. they must be loaded on container ships with respective capacity in order to be transported along the relevant routes. This requirement can be formalized by the following constraint:

\[ \sum_{j=1}^{n} \lambda_{ij} x_{ij} \leq a_{i}, \ (i = \overline{1,m}). \]

Furthermore, it is clear that the container ships of each type to be allocated to the respective routes must be consistent with those that the company actually owns and/or uses. Hence the next constraint:

\[ \sum_{i=1}^{m} x_{ij} = b_{j}, \ (j = \overline{1,n}). \]

On the other hand, the technical capabilities of the individual container ships to carry goods along the respective routes should also be taken into account, i.e.

\[ 0 \leq x_{ij} \leq f_{ij}, \ (i = \overline{1,m}; \ j = \overline{1,n}). \]

Moreover, besides this constraint, the variables \( x_{ij} \) should be positive natural numbers or zeros, given the fact that the object sought is a number of indivisible units such as container ships.

The above brings us to the following economic-mathematical model:

Find the minimum value of the linear form

\[ Z(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + r_{j} d_{ij}) x_{ij} \]  \hspace{1cm} (1)

subject to the following constraints:

\[ \sum_{j=1}^{n} \lambda_{ij} x_{ij} \leq a_{i}, \ (i = \overline{1,m}). \]  \hspace{1cm} (2)

\[ \sum_{i=1}^{m} x_{ij} = b_{j}, \ (j = \overline{1,n}). \]  \hspace{1cm} (2)

\[ 0 \leq x_{ij} \leq f_{ij}, \ (i = \overline{1,m}; \ j = \overline{1,n}). \]  \hspace{1cm} (2)

Moreover, besides this constraint, the variables \( x_{ij} \) should be positive natural numbers or zeros, given the fact that the object sought is a number of indivisible units such as container ships.
The model (1)–(4) may be brought under the class of operational problems related to allocation of resources, i.e. in terms of its structure this model is comparable to the general allocation problem, with the exception of constraint (4) (Atanasov, 2010). The existing constraints (4) and the requirement that the variables be integers makes it considerably harder to solve model (1)–(4). In this regard, the following is an attempt to propose a suitable approach and a corresponding method of finding the optimal plan for the model. To this aim, we shall first formulate and subsequently use another problem, provisionally designated as an evaluation problem. This evaluation problem will be formulated for the general case of the problem of integer optimization (Atanasov et al., 2010):

\[ Z_0 = \max_{X \in K} Z(X), \]  

where \( X = (x_1, x_2, \ldots, x_n) \) is \( n \) – dimensional vector, and \( K \) is an area of permissible solutions where the variables \( x_j, (j = 1, n) \) can only be integers.

**Definition:** The problem \( Z_1 = \max_{X \in R} F(X) \) will be referred to as an evaluation of problem (5), if \( K \subseteq R \) and \( f(X) \leq F(X) \) where \( \forall x \in K \).

Thus, from the definition of evaluation problem directly follows that \( Z_0 \leq Z_1 \) where \( \forall x \in K \).

The evaluation problem is formed on the basis of the Lagrange function. Specifically for the purposes of this study, the formation of the evaluation problem will be shown for a problem of integer linear optimization, and namely:

\[ Z_0 = \max_{X \in K} CX, \]  

where

\[ K = \{ X | AX = A_0, \ X \in M \}, \]

\[ M = \{ X | 0 \leq X \leq D, \ X - \text{integer vector} \} \]

Here, when writing down the linear integer problem (6), the following notations were used:

\[ A = \begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n} \] – constraint matrix;
When we change the order of seeking the maximum and minimum value, we see that:

$$Z_0 \leq \min \Phi_L(U),$$

where

$$\Phi_L(U) = \max_{X \in \mathbb{R}} L \left[ CX + U(A_0 - AX) \right] = \max_{X \in \mathbb{R}} L \left[ (C - UA)X + UA_0 \right].$$

We will refer to the vector $U^*$ for which the function $\Phi_L(U)$ has its maximum value in the area $U \geq 0$ as the vector of Lagrange multipliers (Krass and Chuprinov, 2013). For these multipliers the following inequality is true (Tomlin, 1970):

$$Z_0 \leq \Phi_L(U^*).$$

Before we proceed with construction of the evaluation problem for model (1)-(4), we will look at additional problems complementing the study.
As an example, let us consider a ferry with a total volume of holds and decks $V m^3$ and tonnage $P t$, which can be used to transport $n$ types of load (containers, pallets, etc.). A unit load of the $j$ type has volume $v_j$, weight $p_j$ and value $c_j (j = 1, n)$. It is necessary to select and load those unit loads whose total value is the maximum, taking into account the characteristics of the ferry in terms of capacity and tonnage.

The effectiveness criterion for first problem so formulated is the total value of loads of all kinds with which the ferry must be loaded. Here the control factors are the capacity and tonnage of the ferry, the characteristics of the load (value, volume and weight of unit load). Alternative problems would be such based on other possible combinations of type and quantity of the loads. If the quantities of loads of the $j$ type to be shipped are denoted as $x_j (j = 1, n)$, then the alternatives will be represented as an aggregate of loads to be shipped $\{x_1, x_2, ..., x_n\}$, but only those meeting the restrictions in terms of capacity and tonnage of the ferry.

With such formulation of the problem, its mathematical model will look as follows:

Find the maximum of the function

$$F(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} c_j x_j$$

subject to these constraints:

– capacity

$$\sum_{j=1}^{n} v_j x_j \leq V,$$

– tonnage

$$\sum_{j=1}^{n} p_j x_j \leq P,$$

– variables

$$x_j \geq 0 (j = 1, n) – integers$$

It is quite natural to combine the constraints in terms of capacity and tonnage into one, so that the model now has the following simplified structure:
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\[ \max : F = \sum_{j=1}^{n} c_j x_j \]  \hspace{1cm} (7)  

subject to constrains

\[ \sum_{j=1}^{n} (v_j + p_j) x_j \leq V + P, \]  \hspace{1cm} (8)  
\[ x_j \geq 0 \ (j = \overline{1,n}) \text{ – integers} \]  \hspace{1cm} (9)  

In professional literature the model (7) – (9) is known as the knapsack (back-pack) problem (Pisaruk, 2010). In spite of the fact that special methods of solving have been developed, we will attempt another, slightly different approach to solving this problem.

First of all let’s stipulate that the control parameters \( P > 0, \ V > 0, \ c_j > 0, \ v_j > 0, \ p_j > 0 \ (j = \overline{1,n}) \) are integers. This requirement is fully justified by the existing practice. Let us consider

\[ \frac{c_k}{v_k + p_k} = \max \left\{ \frac{c_j}{v_j + p_j} \ \mid j = \overline{1,n} \right\}. \]  \hspace{1cm} (10)  

We shall assume that not all \( c_j \ (j = \overline{1,n}) \) equal zero, i.e. \( c_k > 0 \).

Theorem: If the parameter \( c_k \) determined through equation (10) is divisor of the numbers \( c_j \ (j = \overline{1,n}) \), then

\[ x_k = \left[ \frac{V + P}{v_k + p_k} \right], \ x_j = 0 \ (j = \overline{1,n}; \ j \neq k) \]  \hspace{1cm} (11)  

is the solution of the problem (7) – (9).

In order to prove this assertion we will examine the equivalent of problem (7) – (9): find the maximum of the function \( \frac{F}{c_k} \), given that:

\[ -\frac{F}{c_k} + \sum_{j=1}^{n} \frac{c_j}{c_k} x_j \leq 0, \]  \hspace{1cm} (12)
On the grounds of (11) and inequality (13) it follows, that all coefficients before the variables are non-negative numbers, which is why

\[
\frac{F}{c_k} = \frac{V + P}{v_k + p_k}, \quad x_j = 0 \quad (j = 1, n; j \neq k)
\]

is obviously solution to problem (12) – (14), and therefore (11) is solution to problem (7) – (9). To summarize, if

\[
\frac{F}{c_k} = \frac{V + P}{v_k + p_k}, \quad x_j = x_j^* \quad (j = 1, n; j \neq k)
\]

is solution to problem (12) – (14), then

\[
x_k = \frac{V + P}{v_k + p_k} - \sum_{j=1}^{n} \frac{c_j}{c_k} x_j^*, \quad x_j = x_j^* \quad (j = 1, n; j \neq k)
\]

will be solution to problem (7) – (9).

The second supplementary problem will be formulated as follows. Suppose that a vessel with tonnage (capacity) \( P \) units must be loaded with \( n \) different types of loads. The aggregate costs \( c_j \) and weight (volume) \( a_j \) of one unit of load of the \( j \) type of load are known. The units of each load are indivisible, i.e. these are containers or other similar transport structures. The load must be assembled on the vessel in such manner so as to minimize the costs for its transport.

If we denote with \( x_j \) the units of the \( j \) type of load, which must be loaded, transported and unloaded from one port to another, then the mathematical model of the problem will look as follows:

Find the minimum of the function

\[
Z = \sum_{j=1}^{n} c_j x_j
\]

subject to the constraints
To find the functional equations of the problem, we assume that we have loaded $x_n$ units of the $n$ type load, where $x_n$ is integer, $0 \leq x_n \leq \left[ \frac{P}{a_n} \right]$. According to Bellman’s Principle of Optimality (Lalov et al., 1973), the remaining tonnage (capacity) $P - a_n x_n$ should be used efficiently. We then introduce the function $f_n(P)$, i.e. the minimum value of transport of type $n$ loads which can be loaded on vessel with tonnage (capacity) $P$.

Thus, the aggregate costs of the transport for solution $x_n$ will be

$$c_n x_n + f_{n-1}(P - a_n x_n).$$

(18)

Since $x_n$ is an integer from the closed interval $\left[ 0; \left[ \frac{P}{a_n} \right] \right]$, then the optimal solution with respect to $x_n$ is the one minimizing the expression (18), i.e.:

$$f_n(P) = \min_{0 \leq x_n \leq \left[ \frac{P}{a_n} \right]} \{c_n x_n + f_{n-1}(P - a_n x_n)\},$$

for $n \neq 1$, and if $n = 1$,

$$f_1(P) = \min \{c_1 x_1\} = c_1 \left[ \frac{P}{a_n} \right].$$

Thus we arrive at a problem which can be solved using the dynamic optimization methods (Spiridonov, 1978).

After this preparation, let us focus on the formulation and solution of the evaluation problem for models (1) – (4). For the purpose of formulation, we will include the constraints (3) in the Lagrange function, while other constraints will remain in the same area where the minimization of the Lagrange function will take place. Thus, for model (1) – (4) in particular, we get:
\[ F = \min_R \max_U \left[ \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + r_j d_{ij})x_{ij} + \sum_{j=1}^{n} u_j (b_j - \sum_{i=1}^{m} x_{ij}) \right] \geq \max_U \Phi(U) = \Phi(U^*), \]

where

\[ R = \{ X \mid \sum_{j=1}^{n} \lambda_{ij} x_{ij} \leq a_i, (i = 1, m), 0 \leq x_{ij} \leq f_{ij} - \text{integers}\}, \]

\[ \Phi(U) = \min_R \left[ \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} + r_j d_{ij})x_{ij} + \sum_{j=1}^{n} u_j (b_j - \sum_{i=1}^{m} x_{ij}) \right] = \sum_{i=1}^{m} \min_{R_i} \sum_{j=1}^{n} (c_{ij} + r_j d_{ij})x_{ij} + \sum_{j=1}^{n} b_j u_j \]

\[ R_i = \{ (x_{i1}, x_{i2}, ..., x_{in}) \mid \sum_{j=1}^{n} \lambda_{ij} x_{ij} \leq a_i, (i = 1, m), 0 \leq x_{ij} \leq f_{ij} - \text{integers}\}. \]

To find the value of the function \( F(U) \) at a given point we need, according to (19), to solve \( m \) “knapsack problems”, i.e. problems such as (7) – (9), for whose linear solution an approach was proposed above.

It should be emphasized that the vector \( U^* \) can be found using the methods of linear programming (Atanasov et al., 2014), where the participating vectors can be constructed in the course of the solution. Indeed, we will note that the set \( R_i \) contains a finite number of points: \( R_i = \{ X_i^t \mid t \in T_i \} \).

We denote:

\[ \mu_i = \min_{R_i} \sum_{j=1}^{n} (c_{ij} + r_j d_{ij} - u_j) x_{ij} = \min_{t \in T_i} \sum_{j=1}^{n} (c_{ij} + r_j d_{ij} - u_j) x_{ij}^t. \]

Then for \( \forall t \in T_i \) the following will be true

\[ \sum_{j=1}^{n} (c_{ij} + r_j d_{ij})x_{ij}^t - \sum_{j=1}^{n} x_{ij}^t u_j \geq \mu_i \]

and the problem of finding the maximum of \( \Phi(U) \) can be represented as follows:

\[ \Phi(U^*) = \max \left( \sum_{i=1}^{m} \mu_i + \sum_{j=1}^{n} b_j u_j \right) \]

\[ \mu_i + \sum_{j=1}^{n} x_{ij}^t u_j \leq \sum_{j=1}^{n} (c_{ij} + r_j d_{ij})x_{ij}^t, t \in T_i, (i = 1, m). \]  

(20)

Then we construct the dual problem:
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\[
\min \sum_{i=1}^{m} \sum_{t \in T_i} \left( (c_{ij} + r_j d_{ij})x_{ij}^t \right) y_{it}, \quad (21)
\]

\[
\sum_{i=1}^{m} \sum_{t \in T_i} x_{ij}^t y_{it} = b_j, \quad (j = 1, n), \quad (22)
\]

\[
\sum_{i \in T_i} y_{it} = 1, \quad (i = 1, m), \quad (23)
\]

\[
y_{it} \geq 0, \quad (i = 1, m; \, t \in T). \quad (24)
\]

The number of vectors of problem (21) – (24), which is the same as the number of points in the set \( R_i \) \( (i = 1, m) \) is obviously too large. But in order to solve problem (21) – (24) it is not necessary to “run along” all vector. As with the decomposition method (Goldstein and Yudin, 1971), it is possible at each step of the simplex algorithm to look for that vector for which the condition of the problem is violated to the greatest degree. Specifically for the variables \( y_{it} \) this condition is determined through (20). Consequently, for each index \( i \) we must find such vector \( X_i^t \), for which the expression

\[
\sum_{j=1}^{n} (c_{ij} + r_j d_{ij})x_{ij}^t - \sum_{j=1}^{n} x_{ij}^t u_j = \sum_{j=1}^{n} (c_{ij} + r_j d_{ij} - u_j)x_{ij}^t
\]

has its minimum value, where \( u_j \) are resolving multipliers corresponding to the constraints (22). Therefore, in order to determine the vectors to be included in the basis of problem (21) – (24), we need to solve \( m \) problems of the type:

\[
\sum_{j=1}^{n} (c_{ij} + r_j d_{ij})x_{ij}^t - \sum_{j=1}^{n} x_{ij}^t u_j = \sum_{j=1}^{n} (c_{ij} + r_j d_{ij} - u_j)x_{ij}^t \quad (25)
\]

\[
\sum_{j=1}^{n} \lambda_{ij} x_{ij} \leq a_i, \quad (26)
\]

\[
0 \leq x_{ij} \leq f_{ij}; \quad x_{ij} \text{ integers for } j = 1, n. \quad (27)
\]

Each of these problems (25) – (27) belongs to the class of the operational problems (7) – (9) discussed above (note though that the index \( i \) is invariable), for whose solution the dynamic programming method was recommended. If \( F_i(U) - \mu_i < 0 \), where \( \mu_i \) is a resolving multiplier corresponding to (23), then the optimality condition (20)
will be violated and the vector \( X_i^t = (x_{i1}^t, x_{i2}^t, \ldots, x_{im}^t, 0, \ldots, 1, 0, \ldots, 0) \) whose components are the solutions of problem (25) – (27) should be introduced to the basis of problem (21) – (24). After a finite number of steps we will arrive at the optimal solution of this problem. The resolving multiplier \( U_j^* \) corresponding to (22) determines the vector whose components are the Lagrange multipliers, and \( F_i = \Phi(U^*) \) determines the evaluation sought.

Overall, it should be noted that model (1) – (4) may in the most general case be defined as a transportation problem. In these problems, as is well known (Zhelevyazkova, 2011), the variables \( x_{ij} \) participate in exactly two of the constraints: they appear once in each. This allows the evaluation problem to be constructed in another manner, for example by including another subsystem in the Lagrange function. In particular, in the case of the examined model (1) – (4) we may include in the Lagrange function the following constraint:

\[
\sum_{j=1}^{n} \lambda_{ij} x_{ij} \leq a_i
\]

and take into account that \( 0 \leq x_{ij} \leq f_{ij} ; \ x_{ij} = \text{are integers. Thus we have}

\[
Z(X) \leq \max_{U \geq 0} \left\{ \sum_{j=1}^{n} \min_{R_j} \left( \sum_{i=1}^{m} R_{ij} \sum_{i=1}^{m} (c_{ij} + r_j d_{ij} + \lambda_{ij} u_j) x_{ij} - \sum_{i=1}^{m} a_i u_i \right) \right\}, \quad (28)
\]

where

\[
R_j = \left\{ x_{ij} \left| \sum_{i=1}^{m} x_{ij} = b_j, 0 \leq x_{ij} \leq f_{ij}, x_{ij} = \text{integers (} j \text{ is invariable)} \right. \right\}.
\]

The optimal solution of the sub-problem to the evaluation problem (28) can easily be found by applying known methods (Kenderov et al., 1989). It should be emphasized that the evaluation determined through (28) coincides with the solution of model (1) – (4).

Indeed, when the set

\[
R_j' = \left\{ x_{ij} \left| \sum_{i=1}^{m} x_{ij} = b_j, 0 \leq x_{ij} \leq f_{ij}, (j \text{ is invariable}) \right. \right\}
\]

is formed and the duality theorem (Atanasov, 2009) is applied, we have
We will further note, that (29) is different from (28) only insofar as when defining the area \( R_j \) the variables \( x_{ij} \) must be integers, while in (29) there is no such requirement.

Regardless, it should be taken into account that the optimal solution of sub-problem (29) also will consist of integers, which ensures the coincidence of (28) with (29).

When looking for Lagrange multipliers for different integer transportation problems, the general scheme for solving the problem of linear programming remains unchanged. Changes occur only to the sub-problems used to determine the basis vectors, as well as to the type of these vectors. In particular, as regards model (1) – (4), the sub-problem performing the function of (25) – (27) will be expressed as follows:

\[
\min \left\{ \sum_{j=1}^{n} (c_{ij} + r_j d_{ij} - u_j) x_{ij} + s_i y_{ij} \mid \sum_{j=1}^{n} x_{ij} = a_i; 0 \leq x_{ij} \leq f_{ij} y_{ij}; \ y_{ij} = y_{ij}^2, \ j = 1, n \right\}. \tag{30}
\]

When \( a_i \) and \( f_{ij} \) are known, an optimal solution for (30) may be sought for integer values of \( x_{ij} \). This allows us to reduce (30) to the generalized knapsack problem (15) – (17) expressed as follows:

\[
\min \left\{ \sum_{j=1}^{n} \sum_{p=1}^{f_{ij}} k^{(i)}_{jp} z^{(i)}_{jp} \mid \sum_{j=1}^{n} \sum_{p=1}^{f_{ij}} p z^{(i)}_{jp} \leq a_i, \sum_{p=1}^{f_{ij}} z^{(i)}_{jp} \leq 1; z^{(i)}_{jp} \in \{0, 1\} \right\}, \tag{31}
\]

where \( z^{(i)}_{jp} = 1 \), if \( x_{ij} = p, y_{ij} = 1 \) and \( z^{(i)}_{jp} = 0 \) in all other cases. Then

\[
k^{(i)}_{jp} = (c_{ij} + r_j d_{ij} - u_j) p + s_{ij}.
\]

Problems (30) and (31) can be solved through the dynamic programming method. It is based on the basic functional equations and recurrent correlations (Spiridonov, 1978). Specifically for the case at hand, given that the index \( i \) is invariable, applied to (25) – (27), it can be described in the following way. Let

\[
\varphi_k l = \min \left\{ \sum_{j=1}^{n} (c_j - u_j) x_j \mid \sum_{j=1}^{n} \lambda_j x_j \leq l, 0 \leq x_j \leq f_j - \text{integers}, (j = 1, n) \right\}.
\]
Then
\[ \varphi_{k+1}(l) = \min_{x_{k+1} = 0, l_{k+1} \leq l} \{(c_{k+1} - u_{k+1})x_{k+1} + \varphi_k(l - l_{k+1}x_{k+1})\}, \tag{32} \]
where \( \varphi_k(l) = \infty \), if \( l < 0 \).

The function \( \varphi_k(l) \) is calculated directly, while \( \varphi_{k+1}(l) \) is derived from (32) where \( k = 1, n - 1 \). The value \( \varphi_n(a) \) determines the minimum value sought, and this process, which is the reverse process to the calculation of the values of \( \varphi_k(l) \), allows finding the optimal solution of the problem. The calculation of the values of \( \varphi_k(l) \), along with the finding of conditionally optimal values can be performed easily by using special tables (Atanasov, 2015).

In conclusion, we should point out that after some adaptation the proposed approach can be successfully applied to solving other types of transportation and logistics problems related to discrete variables. Similarly, the calculation procedure at the different stages of solution of the model (1) – (4) is not associated with major difficulties, since the individual blocks of sub-problems are in fact optimization problems, for which special methods were proposed, and besides these can be dealt with using software.

References


20. [www.zweigmedia.com](http://www.zweigmedia.com), [www.wolframalpha.com](http://www.wolframalpha.com)